

# SUPERCritical NONLINEAR SCHRÖDINGER EQUATIONS I: QUASI-PERIODIC SOLUTIONS

W.-M. WANG

ABSTRACT. We construct time quasi-periodic solutions of arbitrary number of frequencies for arbitrary supercritical algebraic nonlinear Schrödinger equations on the  $d$ -torus ( $d$  arbitrary). In the high frequency limit, these quantitative solutions could also be relevant to compressible Euler equations. The main new ingredient is a *geometric* selection in the Fourier space.

## CONTENTS

1. Introduction and statement of the Theorem
2. The first step in the Newton scheme—  
extraction of parameters
3. The second step
4. Proof of the Theorem
5. Appendix: the cubic nonlinearity

## 1. Introduction and statement of the Theorem

We consider the nonlinear Schrödinger equation on the  $d$ -torus  $\mathbb{T}^d = [0, 2\pi)^d$ :

$$i \frac{\partial}{\partial t} u = -\Delta u + |u|^{2p} u + H(x, u, \bar{u}) \quad (p \geq 1, p \in \mathbb{N}), \quad (1.1)$$

with periodic boundary conditions:  $u(t, x) = u(t, x + 2n\pi)$ ,  $x \in [0, 2\pi)^d$  for all  $n \in \mathbb{Z}^d$ , where  $H(x, u, \bar{u})$  is analytic and has the expansion:

$$H(x, u, \bar{u}) = \sum_{m=1}^{\infty} \alpha_m(x) |u|^{2p+2m} u,$$

with  $\alpha_m$  periodic and uniformly real analytic. The integer  $p$  in (1.1) is *arbitrary*.

Let  $u^{(0)}$  be a solution to the linear equation:

$$i \frac{\partial}{\partial t} u^{(0)} = -\Delta u^{(0)}. \quad (1.2)$$

We seek quasi-periodic solutions to (1.1) with  $b$  frequencies in the form

$$u(t, x) = \sum_{(n, j)} \hat{u}(n, j) e^{in \cdot \omega t} e^{ij \cdot x}, \quad (n, j) \in \mathbb{Z}^{b+d}, \quad (1.3)$$

with  $\omega \in \mathbb{R}^b$  to be determined. Writing in this form, a solution  $u^{(0)}$  to (1.2) with  $b$  frequencies  $\omega^{(0)} = \{j_k^2\}_{k=1}^b$  ( $j_k \neq 0$ ) has Fourier support

$$\text{supp } \hat{u}^{(0)} = \{(-e_{j_k}, j_k), k = 1, \dots, b\}, \quad (1.4)$$

where  $e_{j_k}$  is a unit vector in  $\mathbb{Z}^b$  and  $j_k \neq j_{k'}$  if  $k \neq k'$ . (Unless otherwise stated  $j_k^2 := |j_k|^2$  etc.)

Define the bi-characteristics

$$\mathcal{C} = \{(n, j) \in \mathbb{Z}^{b+d} \mid \pm n \cdot \omega^{(0)} + j^2 = 0\}. \quad (1.5)$$

$\mathcal{C}$  is the solution set in the form (1.3) to (1.2) and its complex conjugate in the Fourier space. Define

$$\begin{aligned} \mathcal{C}^+ &= \{(n, j) \mid n \cdot \omega^{(0)} + j^2 = 0, j \neq 0\} \cup \{(n, 0) \mid n \cdot \omega^{(0)} = 0, n_1 \leq 0\}, \\ \mathcal{C}^- &= \{(n, j) \mid -n \cdot \omega^{(0)} + j^2 = 0, j \neq 0\} \cup \{(n, 0) \mid n \cdot \omega^{(0)} = 0, n_1 > 0\}, \\ \mathcal{C}^+ \cap \mathcal{C}^- &= \emptyset, \quad \mathcal{C}^+ \cup \mathcal{C}^- = \mathcal{C}. \end{aligned} \quad (1.6)$$

Assume  $u^{(0)}$  is *generic*, satisfying the genericity conditions (i-iv) at the end of this section. Here it suffices to mention that the genericity conditions pertain entirely to the Fourier support of  $u^{(0)}$ :  $j = \{j_k\}_{k=1}^b \in (\mathbb{R}^d)^b$  and are determined by the  $|u|^{2p}u$  term in (1.1) only. These conditions are explicit and moreover the non-generic set  $\Omega$  is of codimension 1 in  $(\mathbb{R}^d)^b$ .

The main result is

**Theorem.** *Assume*

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x},$$

*a solution to the linear equation (1.2) is generic and  $a = \{a_k\} \in (0, \delta]^b = \mathcal{B}(0, \delta)$ . There exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exists  $\delta_0 > 0$  and for all  $\delta \in (0, \delta_0)$  a Cantor set  $\mathcal{G}$  with*

$$\text{meas } \{\mathcal{G} \cap \mathcal{B}(0, \delta)\} / \delta^b \geq 1 - C\epsilon^c. \quad (1.7)$$

*For all  $a \in \mathcal{G}$ , there is a quasi-periodic solution of  $b$  frequencies to the nonlinear Schrödinger equation (1.1):*

$$u(t, x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(\delta^3), \quad (1.8)$$

*with basic frequencies  $\omega = \{\omega_k\}$  satisfying*

$$\omega_k = j_k^2 + \mathcal{O}(\delta^{2p}).$$

*The remainder  $\mathcal{O}(\delta^3)$  is in an analytic norm about a strip of width  $\mathcal{O}(1)$  on  $\mathbb{T}^{b+d}$ .*

*Remark.* The above theorem also holds when there is in addition an overall phase  $m \neq 0$ , corresponding to adding  $m$  to the right side of (1.1). When  $d = p = 1$ , the non-generic set  $\Omega = \emptyset$ . All  $u^{(0)}$  are generic and only amplitude selection is necessary. This is the well understood scenario, see the Appendix in sect. 5. To understand the substance of the geometric and amplitude excisions in the Theorem, it is useful to take  $H = 0$  and note the perpetual existence of periodic solutions:

$$u = a e^{-i(j^2 + |a|^{2p})t} e^{ij \cdot x}$$

to (1.1) for all  $j \in \mathbb{Z}^d$  and  $a \in \mathbb{C}$ .

In  $d \geq 3$  and specializing to  $H = 0$ , (1.1) is supercritical for  $p$  sufficiently large, namely  $p \geq \frac{2}{d-2}$  ( $d \geq 3$ ), where the local well-posedness is in  $H^s$  for  $s > 1$ . This is above the Hamiltonian topology and there are no global existence results [B1]. The proceeding Theorem uses a Newton scheme to construct a type of global solutions with precise control over the Fourier coefficients. Moreover we have the following semi-classical analog, which is new to the KAM context:

**Corollary.** *Set  $H = 0$  in (1.1). Assume*

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x},$$

*a solution to the linear equation (1.2) is generic,  $\{j_k\}_{k=1}^b \in [K\mathbb{Z}^d]^b$ ,  $K \in \mathbb{N}^+$  and  $a = \{a_k\} \in (0, 1]^b = \mathcal{B}(0, 1)$ . There exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exists  $K_0 > 0$  and for all  $K > K_0$  a Cantor set  $\mathcal{G}$  with*

$$\text{meas } \{\mathcal{G} \cap \mathcal{B}(0, 1)\} \geq 1 - C\epsilon^c.$$

For all  $a \in \mathcal{G}$ , there is a quasi-periodic solution of  $b$  frequencies to the nonlinear Schrödinger equation (1.1):

$$u(t, x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(1/K^2),$$

with basic frequencies  $\omega = \{\omega_k\}$  satisfying

$$\omega_k = j_k^2 + \mathcal{O}(1).$$

The remainder  $\mathcal{O}(1/K^2)$  is in an analytic norm about a strip of width  $\mathcal{O}(1)$  in  $t$  and  $\mathcal{O}(1/K)$  in  $x$  on  $\mathbb{T}^{b+d}$ .

*Remark.* These are quantitative global  $\mathbb{L}^2$  size 1 and large (kinetic) energy solutions, which could be relevant to compressible Euler equations in the low density limit, cf. [G, Se].

In part II, using a related finitely iterated Newton scheme we construct analytic solutions for *fixed generic* initial conditions and prove almost global existence for Cauchy problems. This is reasonable as the known invariant measure for smooth flow is supported by KAM tori.

*A sketch of proof of the Theorem*

We write (1.1) in the Fourier space, it becomes

$$\text{diag } (n \cdot \omega + j^2) \hat{u} + (\hat{u} * \hat{v})^{*p} * \hat{u} + \sum_{m=1}^{\infty} \hat{\alpha}_m * (\hat{u} * \hat{v})^{*(p+m)} * \hat{u} = 0, \quad (1.9)$$

where  $(n, j) \in \mathbb{Z}^{b+d}$ ,  $\hat{v} = \hat{u}$ ,  $\omega \in \mathbb{R}^b$  is to be determined and

$$|\hat{\alpha}_m(\ell)| \leq C' e^{-c'|\ell|} \quad (C', c' > 0)$$

for all  $m$ . From now on we work with (1.9), for simplicity we drop the hat and write  $u$  for  $\hat{u}$  and  $v$  for  $\hat{v}$  etc. We seek solutions close to the linear solution  $u^{(0)}$  of  $b$  frequencies,  $\text{supp } u^{(0)} = \{(-e_{j_k}, j_k), k = 1, \dots, b\}$ , with frequencies  $\omega^{(0)} = \{j_k^2\}_{k=1}^b$  ( $j_k \neq 0$ ) and small amplitudes  $a = \{a_k\}_{k=1}^b$  satisfying  $\|a\| = \mathcal{O}(\delta) \ll 1$ .

We complete (1.9) by writing the equation for the complex conjugate. So we have

$$\begin{cases} \text{diag } (n \cdot \omega + j^2) u + (u * v)^{*p} * u + \sum_{m=1}^{\infty} \alpha_m * (u * v)^{*(p+m)} * u = 0, \\ \text{diag } (-n \cdot \omega + j^2) v + (u * v)^{*p} * v + \sum_{m=1}^{\infty} \alpha_m * (u * v)^{*(p+m)} * v = 0, \end{cases} \quad (1.10)$$

By supp, we will always mean the Fourier support, so we write  $\text{supp } u^{(0)}$  for  $\text{supp } \hat{u}^{(0)}$  etc. Let

$$\mathcal{S} = \text{supp } u^{(0)} \cup \text{supp } \bar{u}^{(0)}. \quad (1.11)$$

Denote the left side of (1.10) by  $F(u, v)$ . We make a Lyapunov-Schmidt decomposition into the  $P$ -equations:

$$F(u, v)|_{\mathbb{Z}^{b+d} \setminus \mathcal{S}} = 0,$$

and the  $Q$ -equations:

$$F(u, v)|_{\mathcal{S}} = 0.$$

We seek solutions such that  $u|_{\mathcal{S}} = u^{(0)}$ . The  $P$ -equations are infinite dimensional and determine  $u$  in the complement of  $\text{supp } u^{(0)}$ ; the  $Q$ -equations are  $2b$  dimensional and determine the frequency  $\omega = \{\omega_k\}_{k=1}^b$ .

This Lyapunov-Schmidt method was introduced by Craig and Wayne [CW] to construct periodic solutions for the wave equation in one dimension. It was inspired by the multiscale analysis of Fröhlich and Spencer [FS]. The construction was further developed by Bourgain to embrace the full generality of quasi-periodic solutions and in arbitrary dimensions  $d$  [B2, 3]. More recently, Eliasson and Kuksin [EK] developed a KAM theory in the Schrödinger context. All the above results, however, pertain to parameter dependent tangentially non resonant equations.

We use a Newton scheme to solve the  $P$ -equations, with  $u^{(0)}$  as the initial approximation. The major difference with [CW, B2, 3, EK] is that (1.10) is completely resonant and there are *no* parameters at this initial stage. The frequency  $\omega^{(0)}$  is an *integer* in  $\mathbb{Z}^b$ . So we need to proceed differently.

First recall the formal scheme: the first correction

$$\Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} - \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix} = [F'(u^{(0)}, v^{(0)})]^{-1} F(u^{(0)}, v^{(0)}), \quad (1.12)$$

where  $\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}$  is the next approximation and  $F'(u^{(0)}, v^{(0)})$  is the linearized operator on  $\ell^2(\mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d})$

$$F' = D + A, \quad (1.13)$$

where

$$D = \begin{pmatrix} \text{diag } (n \cdot \omega + j^2) & 0 \\ 0 & \text{diag } (-n \cdot \omega + j^2) \end{pmatrix} \quad (1.14)$$

and

$$\begin{aligned} A &= \begin{pmatrix} (p+1)(u * v)^{*p} & p(u * v)^{*p-1} * u * u \\ p(u * v)^{*p-1} * v * v & (p+1)(u * v)^{*p} \end{pmatrix} + \mathcal{O}(\delta^{2p+2}) \quad (p \geq 1), \\ &= A_0 + \mathcal{O}(\delta^{2p+2}). \end{aligned} \quad (1.15)$$

with  $\omega = \omega^{(0)}$ ,  $u = u^{(0)}$  and  $v = v^{(0)}$ .

Since we look at small data,  $\|A\| = \mathcal{O}(\delta^{2p}) \ll 1$  and the diagonal:  $\pm n \cdot \omega + j^2$  are integer valued, using the Schur complement reduction [S1, 2], the spectrum of  $F'$  around 0 is equivalent to that of a reduced operator on  $\ell^2(\mathcal{C})$ , where  $\mathcal{C}$  is defined in (1.5) and to  $\mathcal{O}(\delta^{2p+2})$  it is the same as the spectrum of  $A_0$  on  $\ell^2(\mathcal{C})$ .

*The genericity conditions*

To define generic  $u^{(0)}$ , we need to analyze the convolution matrix  $A_0$ . Let

$$\Gamma^{++} = \text{supp } [|u^{(0)}|^{2p}] = \{(\Delta n, \Delta j)\},$$

where

$$\begin{aligned}
\Delta n &= - \sum p_{kk'}(e_k - e_{k'}), \\
\Delta j &= \sum p_{kk'}(j_k - j_{k'}), \\
p_{kk'} &\geq 0, \sum p_{kk'} \leq p, k, k' = 1, \dots, b, \\
\{(-e_k, j_k)\}_{k=1}^b &= \text{supp } u^{(0)}, j_k \neq j_{k'} \text{ if } k \neq k',
\end{aligned} \tag{1.16}$$

and

$$\Gamma^{+-} = \text{supp } [|u^{(0)}|^{2(p-1)} u^{(0)2}] = \{(\Delta n, \Delta j)\},$$

where

$$\begin{aligned}
\Delta n &= - \sum p_{kk'}(e_k - e_{k'}) - (e_\kappa + e_{\kappa'}), \\
\Delta j &= \sum p_{kk'}(j_k - j_{k'}) + (j_\kappa + j_{\kappa'}), \\
p_{kk'} &\geq 0, \sum p_{kk'} \leq p - 1, k, k', \kappa, \kappa' = 1, \dots, b,
\end{aligned} \tag{1.17}$$

and

$$\Gamma^{-+} = \text{supp } [|u^{(0)}|^{2(p-1)} v^{(0)2}]. \tag{1.18}$$

Let

$$\mathcal{A} = \cup \prod_{d+2} \Gamma, \tag{1.19}$$

where  $\Gamma = \Gamma^{++}, \Gamma^{+-}$  or  $\Gamma^{-+}$ , the difference of the number of factors of  $\Gamma^{+-}$  and  $\Gamma^{-+}$  in the  $(d+2)$ -fold product is at most 1 and the union is over all such possible choices of  $\Gamma$ . Elements of  $\mathcal{A}$  are of the form:

$$\mathcal{A} \ni (\Delta n, \Delta j) = \sum_{i \leq d+2} ((\Delta n^{(i)}, \Delta j^{(i)}), \tag{1.20}$$

where  $(\Delta n^{(i)}, \Delta j^{(i)}) \in \Gamma^{++}, \Gamma^{+-}$  or  $\Gamma^{-+}$ .

Let  $\sigma$  be a set of  $d+1$  elements in  $\mathcal{A}$ :  $\sigma \subset \mathcal{A} \setminus (0, 0)$ ,  $|\sigma| = d+1$  and

$$\sigma \subset \prod_{d+1} \Gamma^{++}. \tag{1.21}$$

Define

$$J = |\Delta j|^2 + \Delta n \cdot \omega^{(0)}, \tag{1.22}$$

for  $(\Delta n, \Delta j) \in \sigma$ .

Otherwise let  $\sigma$  be a set of  $d+2$  elements in  $\mathcal{A} \setminus (0, 0)$ , such that  $\sigma$  does not contain a  $d+1$  element subset

$$\sigma' \subset \prod_{d+1} \Gamma^{++}. \tag{1.23}$$

Since (1.23) is violated,

$$\sigma \cap \left( \prod_d \Gamma^{++} \right) \Gamma^{+-} \neq \emptyset, \tag{1.24}$$

or

$$\sigma \cap \left( \prod_d \Gamma^{++} \right) \Gamma^{-+} \neq \emptyset. \quad (1.25)$$

From symmetry it suffices to consider  $\sigma$  such that (1.24) holds.

Define

$$\mathbb{A} = \left( \prod_d \Gamma^{++} \right) \Gamma^{+-}.$$

From (1.24)

$$\sigma \cap \mathbb{A} \neq \emptyset. \quad (1.26)$$

Let

$$(a, a') \in \sigma \cap \mathbb{A}. \quad (1.27)$$

On  $\sigma \setminus (a, a')$  define

$$J = |\Delta j|^2 + 2a' \cdot \Delta j - \Delta n \cdot \omega^{(0)}, \quad (1.28)$$

if the sum in (1.20) contains odd number of  $(\Delta n^{(i)}, \Delta j^{(i)}) \in \Gamma^{+-}$ ; otherwise define  $J$  as in (1.22).

*Definition.*  $u^{(0)}$  of  $b$  frequencies is *generic* if its Fourier support  $\{(-e_m, j_m)\}_{m=1}^b$ , where  $j_k \neq j_{k'}$  if  $k \neq k'$  satisfies:

(i) For all  $(\Delta n, \Delta j) \in \mathcal{A} \setminus (0, 0)$ ,

$$\Sigma_{\pm} = |\Delta j|^2 \pm \Delta n \cdot \omega^{(0)} \neq 0,$$

where  $\omega^{(0)} = \{j_k^2\}_{k=1}^b$ .

(ii) For any  $\sigma \in \mathcal{A}$  with  $|\sigma| = d + 1$  satisfying (1.21) and  $\Delta j \neq 0$  identically for any  $(\Delta n, \Delta j) \in \sigma$ , if there exists  $\Delta j \notin \{j_k - j_{k'}, k, k' = 1, \dots, b, k \neq k'\}$ , the  $(d+1) \times (d+1)$  determinant

$$D = \det(2\Delta j, J) \neq 0,$$

where  $J$  as defined in (1.22).

(iii) For any  $\sigma \in \mathcal{A}$  with  $|\sigma| = d + 2$  satisfying (1.26) and  $\Delta j \neq 0$  identically for any  $(\Delta n, \Delta j) \in \sigma$ , if either  $a'$  in (1.27)  $\notin \{j_k + j_{k'}, k, k' = 1, \dots, b\}$  or there exists  $\Delta j \notin \{j_k - j_{k'}, k, k' = 1, \dots, b, k \neq k'\}$ , the  $(d+1) \times (d+1)$  determinant

$$D = \det(2\Delta j, J) \neq 0,$$

where  $J$  as defined in (1.22, 1.28).

(iv) For all  $j_m$ ,  $m = 1, \dots, b$  and all  $(\Delta n, \Delta j) \in \Gamma^{++}$ , the functions

$$f = \Delta n \cdot \omega^{(0)} + 2j_m \cdot \Delta j + |\Delta j|^2 \neq 0,$$

if  $(\Delta n, \Delta j) \neq (-e_{k'} + e_m, j_{k'} - j_m)$ , for all  $k' = 1, \dots, b$ .

Let  $(j_k - j_{k'})$ ,  $k \neq k'$  be a factor present in  $(\Delta n, \Delta j) \in \mathcal{A} \setminus (0, 0)$ , cf. (1.19, 1.16, 1.17). We note that  $\partial \Sigma_{\pm}$  and  $\partial f$  are not identically zero, where  $\partial$  is the directional derivative in  $(j_k - j_{k'})$ . Therefore  $\{\Sigma_{\pm} = 0\}$  and  $\{f = 0\}$  are sets of codimension 1 in  $(\mathbb{R}^d)^b$ .

It follows from the restrictions on  $\Delta j$  and  $a'$  in (ii, iii) that there is  $\Delta j$  such that when setting  $\Delta j = 0$ , the corresponding  $J \neq 0$  identically, where we also used (i). So the quadratic  $J$  is not reducible and  $D$  is not identically 0. Therefore  $\{D = 0\}$  gives a set of codimension 1 in  $(\mathbb{R}^d)^b$ . Combining the above deliberations, we have

**Lemma.** *The non-generic set*

$$(\mathbb{R}^d)^b \supset \Omega := \{\Sigma_{\pm} = 0\} \cup \{D = 0\} \cup \{f = 0\}$$

*has codimension 1.*

Below we briefly indicate the considerations that led to (i-iv). For more details, see sect. 2.

*Origins of the genericity conditions*

To implement the Newton scheme using (1.12), we need to bound  $A_0^{-1}$ . From previous considerations, it suffices to consider  $A_0$  restricted to  $\mathcal{C}$ . For  $u^{(0)}$  satisfying (i-iii), we show in sect. 2 that  $A_0|_{\mathcal{C}} = \oplus \mathcal{A}_0$ , where  $\mathcal{A}_0$  are Töplitz matrices of sizes at most  $(2b + d) \times (2b + d)$ . This can be seen by considering connected sets on  $\mathcal{C}$ .

Assume  $(n, j) \in \mathcal{C}^+$  is connected to  $(n', j') \in \mathcal{C}$  by the convolution operator  $A_0$ , then  $n' = n + \Delta n$  and  $j' = j + \Delta j$ , where  $(\Delta n, \Delta j) \in \text{supp } (u^{(0)} * v^{(0)})^{*p}$ , if  $(n', j') \in \mathcal{C}^+$  and

$$\begin{cases} (n \cdot \omega^{(0)} + j^2) = 0, \\ (n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0; \end{cases} \quad (1.29)$$

and if  $(n', j') \in \mathcal{C}^-$ , then  $(\Delta n, \Delta j) \in \text{supp } (u^{(0)} * v^{(0)})^{*p-1} * u^{(0)} * u^{(0)}$  and

$$\begin{cases} (n \cdot \omega^{(0)} + j^2) = 0, \\ -(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0. \end{cases} \quad (1.30)$$

(Clearly the situation is similar if  $(n, j) \in \mathcal{C}^-$ .)

(1.29, 1.30) define a system of polynomial equations. For  $u^{(0)}$  satisfying (i-iii), we show in sect. 2 that the largest connected set is of size at most  $\max(2b, d+2) \leq 2b+d$ . The connected sets of sizes at most  $2b$  result from translation invariance. The other connected sets are of sizes at most  $d+2$ . The translation invariant sets correspond to degeneracy and are in fact the reason for requiring the leading nonlinear  $\mathcal{O}(\delta^{2p+1})$  term in (1.1) to be independent of  $x$ . The  $x$  dependence of the higher order terms do not matter as they are treated as perturbations.

The invertibility of  $A_0$  is then ensured by making an initial excision in  $a$  as 0 is typically *not* an eigenvalue of a finite matrix. So  $\|F'^{-1}\| \asymp \|A_0^{-1}\| \leq \mathcal{O}(\delta^{-2p})$ . Let

$$F_0(u^{(0)}, v^{(0)}) = \begin{pmatrix} (u^{(0)} * v^{(0)})^{*p} * u^{(0)} \\ (u^{(0)} * v^{(0)})^{*p} * v^{(0)} \end{pmatrix}. \quad (1.31)$$

By requiring

$$\text{supp } F_0(u^{(0)}, v^{(0)}) \cap \{\mathcal{C} \setminus \mathcal{S}\} = \emptyset,$$

which amounts to condition (iv), we obtain from (1.12)

$$\|\Delta u^{(1)}\| = \|\Delta v^{(1)}\| \leq \mathcal{O}(\delta^3)$$



for small  $\delta$ . Inserting this into the  $Q$ -equations, which determine  $\omega$ , we achieve amplitude-frequency modulation:

$$\begin{aligned}\|\Delta\omega^{(1)}\| &\asymp \mathcal{O}(\delta^{2p}) \\ |\det(\frac{\partial\omega^{(1)}}{\partial a})| &\asymp \mathcal{O}(\delta^{2p-1}) > 0\end{aligned}$$

ensuring transversality and moreover Diophantine  $\omega^{(1)}$  on a set of  $a$  of positive measure. The tangentially non resonant perturbation theory in [B2, 3] becomes available.

The first iteration is therefore the key step and is the core of the present construction, which is summarized in Proposition 3.2. The main new ingredient is the fine analysis of resonances via systems of polynomial equations, which provides the geometry to achieve modulated Diophantine frequency as input for the analysis part of the construction.

Previously, quasi-periodic solutions were constructed using partial Birkhoff normal forms for the resonant Schrödinger equation in the presence of the cubic nonlinearity in dimensions 1 and 2 [GXY, KP]. These constructions rely on the specifics of the resonance geometry given by the cubic nonlinearity, see the Appendix in sect. 5.

This paper is the nonlinear component of the resonant perturbation theory. For the linear theory, see [W1, 2].

## 2. The first step in the Newton scheme— extraction of parameters

Since we seek solutions with small amplitude  $a = \{a_k\}_{k=1}^b$ , it is convenient to rescale:  $a_k \rightarrow \delta a_k$  ( $0 < \delta \ll 1$ ). We therefore solve instead:

$$\begin{cases} \text{diag}(n \cdot \omega + j^2)u + \delta^{2p}(u * v)^{*p} * u + \sum_{m=1}^{\infty} \delta^{2p+2m} \alpha_m * (u * v)^{*(p+m)} * u = 0, \\ \text{diag}(-n \cdot \omega + j^2)v + \delta^{2p}(u * v)^{*p} * v + \sum_{m=1}^{\infty} \delta^{2p+2m} \alpha_m * (u * v)^{*(p+m)} * v = 0, \end{cases} \quad (2.1)$$

with  $u|_{\text{supp } u^{(0)}} = u^{(0)} = a \in (0, 1]^b = \mathcal{B}(0, 1)$  and the same for  $v$ . The frequency  $\omega = \omega(a)$  is to be determined by the  $Q$ -equations with the initial approximation  $\omega^{(0)} = \{j_k^2\}_{k=1}^b$  ( $j_k \neq 0$ ).

We use the usual procedure of iteratively solving the  $P$  and then the  $Q$ -equations. The main novelty here is the first step, where  $\omega = \omega^{(0)}$  is a fixed integer (vector), the opposite of a Diophantine vector. So the usual Diophantine separation is not available. We overcome the difficulty by bound the size of connected sets on the bi-characteristics  $\mathcal{C}$  using systems of polynomial equations defined in (1.29, 1.30). The size of the connected set is bounded above by the size of the system where there is a solution.

After the first step, with appropriate restrictions on  $a$ ,  $\omega = \omega(a)$  becomes Diophantine from amplitude-frequency modulation and the non resonant mechanism in [B3] is applicable.

*The first iteration.*

Using  $u^{(0)}$ ,  $v^{(0)}$  and  $\omega^{(0)}$  as the initial approximation, we first solve the  $P$ -equations. This requires an estimate on the inverse of the linearized operator  $F'(u^{(0)}, v^{(0)})$  evaluated at  $\omega = \omega^{(0)}$ . This section is devoted to prove the following.

**Lemma 2.1.** Assume  $u^{(0)} = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x}$  a solution to the linear equation with  $b$  frequencies satisfies (i-iii) and  $a = \{a_k\} \in (0, 1]^b = \mathcal{B}(0, 1) = \mathcal{B} \subset \mathbb{R}^b \setminus \{0\}$ . There exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exist  $\delta_0 > 0$  and  $\mathcal{B}_\epsilon$  with

$$\text{meas}(\mathcal{B}_\epsilon \cap \mathcal{B}) < C\epsilon^c.$$

If  $a \in \mathcal{B} \setminus \mathcal{B}_\epsilon$ , then for all  $\delta \in (0, \delta_0)$ ,

$$\|[F'(u^{(0)}, v^{(0)})]^{-1}\| \leq \mathcal{O}(\delta^{-2p}) \quad (2.2)$$

and there exists  $\beta \in (0, 1)$  such that

$$|[F'(u^{(0)}, v^{(0)})]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (2.3)$$

for all  $|x - y| > 1/\beta^2$ .

*Proof of Lemma 2.1.* Let  $P_\pm$  be the projection on  $\mathbb{Z}^{b+d}$  onto  $\mathcal{C}^\pm$  defined in (1.6),

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} \quad (2.4)$$

on  $\mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d}$  and  $P^c$  the projection onto the complement. The linearized operator  $F'(u^{(0)}, v^{(0)})$  is  $F' = D + A$  with  $D$  as in (1.14) and  $A = \delta^{2p} A_0 + \mathcal{O}(\delta^{2p+2})$  as in (1.15).

From the Schur complement reduction [S1, 2],  $\lambda \in \sigma(F') \cap [-1/2, 1/2]$  if and only if  $0 \in \sigma(\mathcal{H})$ , where

$$\mathcal{H} = PF'P - \lambda + PF'P^c(P^cF'P^c - \lambda)^{-1}P^cF'P \quad (2.5)$$

is the effective operator acting on the bi-characteristics  $\mathcal{C}$ . Since  $\|P^cF'P^c\| > 1 - \mathcal{O}(\delta^{2p}) > 1/2$  and  $\|PF'P^c\| = \mathcal{O}(\delta^{2p})$ , the last term in (2.5) is of order  $\delta^{4p}$ , uniformly for  $\lambda \in [-1/2, 1/2]$ .

So

$$\mathcal{H} = PF'_0P - \lambda + \mathcal{O}(\delta^{2p+2}) \quad (2.6)$$

in  $L^2$  uniformly for  $\lambda \in [-1/2, 1/2]$ , where  $F_0$  as defined in (1.18). To obtain (2.2), it suffices to prove  $\|[PF'_0P]^{-1}\| \leq \mathcal{O}(\delta^{-2p})$ . Since  $PF'_0P = \delta^{2p}PA_0P$ , where  $A_0$  as in (1.15), let  $A' = PA_0P$ , it is important to note here that  $A' = A'(a)$  depends on  $a$  but is independent of  $\delta$ .

*Size of connected sets on the bi-characteristics.*

Given  $x, y \in \mathcal{C}$ , we say  $x$  is connected to  $y$  if  $A_0(x, y) \neq 0$ . Given a set  $\mathcal{S} \subseteq \mathcal{C}$ , we say  $\mathcal{S}$  is connected, if for all  $x \in \mathcal{S}$ ,  $\exists y \in \mathcal{S}$ ,  $y \neq x$  and  $A_0(x, y) \neq 0$ . Let  $\mathcal{S}$  be a connected set on  $\mathcal{C}$ . Assume  $|\mathcal{S}| > d + 2$ , so it contains a connected subset  $S \subseteq \mathcal{S}$  with  $|S| - 2 = d + 1$ .  $S$  can be written as  $S = S^+ \cup S^-$ ,  $S^+ \cap S^- = \emptyset$ ,  $S^+ \subset \mathcal{C}^+$  and  $S^- \subset \mathcal{C}^-$ .

Without loss, we may assume  $|S^+| \geq |S^-|$ . (The other case works the same way using symmetry.) Let  $(n, j) \in S^+$  and  $(n + a, j + a') \in S^-$  with  $(a, a') \in \prod_d \Gamma^{++} \Gamma^{+-}$ ,

if  $S^- \neq \emptyset$ . The connected set  $S$  gives a system of  $|S|$  quadratic (in  $j$ ) polynomials of the form:

$$\begin{cases} (n \cdot \omega^{(0)} + j^2) = 0, \\ \pm(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 = 0; \end{cases} \quad (2.7)$$

where  $(\Delta n, \Delta j) \in \prod_{d+2} \Gamma$  and  $\Gamma = \Gamma^{++}, \Gamma^{+-}$  or  $\Gamma^{-+}$ .

There are  $|S^+|$  polynomials corresponding to the points on  $\mathcal{C}^+$  and  $|S^-|$  polynomials for points on  $\mathcal{C}^-$ . Subtracting pairwise the polynomial equations for points on  $\mathcal{C}^+$  from (2.7), we obtain a system of  $|S^+| - 1$  linear equations of the form:

$$2j \cdot \Delta j + |\Delta j|^2 + \Delta n \cdot \omega^{(0)} = 0; \quad (2.9)$$

subtracting pairwise the polynomials on  $\mathcal{C}^-$  from

$$-(n + a) \cdot \omega + (j + a')^2 = 0, \quad (2.10)$$

we obtain a system of  $|S^-| - 1$  linear equations of the form:

$$2(j + a') \cdot \Delta j + |\Delta j|^2 - \Delta n \cdot \omega^{(0)} = 0. \quad (2.11)$$

If some  $\Delta j = 0$  identically, condition (i) gives  $\Delta n \cdot \omega^{(0)} \neq 0$ . So (2.9) or (2.11) is violated for such  $\Delta j$  and  $|S| \leq d + 2$ . Below we assume none of the  $\Delta j = 0$  identically.

Let

$$J = |\Delta j|^2 + \Delta n \cdot \omega^{(0)}, \quad (2.12)$$

or

$$J = |\Delta j|^2 + 2a' \cdot \Delta j - \Delta n \cdot \omega^{(0)}. \quad (2.13)$$

If  $|S^-| \leq 1$ , then in (2.14) below  $J$  is defined only by (2.12), otherwise it is defined either by (2.12) or (2.13).

(\*) Either  $a'$  in (2.11)  $\notin \{j_k + j_{k'}, k, k' = 1, \dots, b\}$  or there exists  $\Delta j$  in (2.9, 2.11)

$$\notin \{j_k - j_{k'}, k, k' = 1, \dots, b, k \neq k'\}.$$

Then from (ii, iii) the  $(d + 1) \times (d + 1)$  determinant:

$$\det(2\Delta j, J) \neq 0, \quad (2.14)$$

where  $\Delta j$  is such that  $(\Delta n, \Delta j) \in \prod_{d+2} \Gamma$ . This gives that if the linear system in (2.9, 2.11) for  $S$  is larger than  $d$ , then it has no solution and if it has  $d$  equations then it has at most 1 solution. So for generic  $u^{(0)}$ ,  $|S^+| + |S^-| - 2 \leq d$  leading to  $|S| \leq d + 2$ .

(\*\*) (\*) is violated.

In this case, (2.14)=0 identically, the solution to (2.9) has  $j$  coordinates in  $\{-j_k, k = 1, \dots, b\}$ , corresponding to the connected set of at most  $2b$  points:  $\{\pm j_k, k = 1, \dots, b\}$ . It is easy to verify that  $\Delta j = 0$  if and only if  $\Delta n = 0$ .

Combining  $(*, **)$ , we obtain  $|S| \leq \max(2b, d+2) \leq 2b+d$ . So  $|S| \leq 2b+d$ .

In conclusion, the above argument gives

$$A'(a) = \oplus A'_\alpha(a), \quad (2.15)$$

where  $\alpha$  are connected sets on  $\mathcal{C}$  and  $A'_\alpha$  are matrices of sizes at most  $(2b+d) \times (2b+d)$ . Since  $A_0(a)$  is a convolution matrix and  $A_0(x, y) = A_0(x-y, 0) \neq 0$  for at most  $2b^{2p}$  of  $(x-y)$ , there are at most  $2^{2b^{2p}} = 4^{b^{2p}} = K$  types of  $A'_\alpha$ , which we rename as  $\mathcal{A}_k$ ,  $1 \leq k \leq K$ .

For each  $k$ ,  $\det \mathcal{A}_k(a) = P_k(a)$  is a polynomial in  $a$  of degree at most  $2p(2b+d)$ . For all  $k$ ,  $P_k$  is a non constant function on  $\mathcal{B} = (0, 1]^b$ , which can be seen as follows. Set  $a = (a_1, 0, \dots, 0)$ . From the structure of  $A_0$  in (1.15),  $\mathcal{A}_k$  is then either a diagonal matrix with diagonal elements  $(p+1)|a_1|^{2p}$  or a matrix with diagonal elements  $(p+1)|a_1|^{2p}$  and 2 non zero off diagonal elements  $p|a_1|^{2(p-1)}a_1^2$  and  $p|a_1|^{2(p-1)}\bar{a}_1^2$ . The determinant  $P_k$  is therefore not a constant. So there exist  $C, c > 0$ , such that for all  $0 < \epsilon < 1$ ,

$$\text{meas } \{a \in \mathcal{B} \mid |P_k| < \epsilon, \text{ all } k \leq K\} \leq C\epsilon^c, \quad (2.16)$$

(cf. e.g., [Lemma 11.4, GS] and references therein).

Since  $\|\mathcal{A}_k\| \leq \mathcal{O}(1)$ , if  $|\det \mathcal{A}_k| > \epsilon$ , then  $\|[\mathcal{A}_k]^{-1}\| \leq \mathcal{O}(\epsilon^{-1})$ . (The exponent is 1 because of self-adjointness.) This proves (2.2).

To prove (2.3), we use the resolvent expansion:

$$[F']^{-1} = [\tilde{F}]^{-1} + [\tilde{F}]^{-1}\Gamma[F']^{-1},$$

where

$$\begin{aligned} \tilde{F} &= \oplus_\alpha A'_\alpha \oplus_{(n,j) \notin \mathcal{C}} \text{diag } (\pm n \cdot \omega^{(0)} + j^2) \\ \Gamma &= F' - \tilde{F} = F'_0 - \tilde{F} + \mathcal{O}(\delta^{2p+2}) = \Gamma_1 + \mathcal{O}(\delta^{2p+2}). \end{aligned}$$

Now

$$\begin{aligned} \|\tilde{F}^{-1}\| &= \mathcal{O}(\delta^{-2p}), \text{ and} \\ \|([\tilde{F}]^{-1}\Gamma_1)^{2m}\| &\leq \mathcal{O}(\delta^{2pm}) \end{aligned} \quad (2.17)$$

for  $m = 1, 2, \dots$ , since  $[A'_\alpha]^{-1}\Gamma_1[A'_{\alpha'}]^{-1} = 0$ , which is obvious when  $\alpha = \alpha'$  and when  $\alpha \neq \alpha'$  from the fact that they are not connected by  $\Gamma_1$ . (2.2, 2.17) give (2.3) for some  $\beta > 0$ .  $\square$

Using Lemma 2.1 to solve the  $P$  and then the  $Q$ -equations, we obtain the following result after the first iteration. Let

$$\Delta u^{(1)} = u^{(1)} - u^{(0)}, \Delta v^{(1)} = v^{(1)} - v^{(0)}, \Delta \omega^{(1)} = \omega^{(1)} - \omega^{(0)}.$$

**Proposition 2.2.** Assume  $u^{(0)} = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x}$  a solution to the linear equation with  $b$  frequencies is generic and  $a = \{a_k\} \in (0, 1]^b = \mathcal{B}(0, 1) = \mathcal{B} \subset \mathbb{R}^b \setminus \{0\}$ . There exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exist  $\delta_0 > 0$  and  $\mathcal{B}_\epsilon$  with

$$\text{meas}(\mathcal{B}_\epsilon \cap \mathcal{B}) < C\epsilon^c.$$

If  $a \in \mathcal{B} \setminus \mathcal{B}_\epsilon$ , then for all  $\delta \in (0, \delta_0)$ ,

$$\|\Delta u^{(1)}\| = \|\Delta v^{(1)}\|_{\ell^2(\rho)} = \mathcal{O}(\delta^3), \quad (2.18)$$

where  $\rho$  is a weight on  $\mathbb{Z}^{b+d}$  satisfying

$$\begin{aligned} \rho(x) &= e^{\beta|x|}, \quad 0 < \beta < 1 \text{ for } |x| > x_0 \\ &= 1, \quad \text{for } |x| \leq x_0. \end{aligned}$$

$$\|F(u^{(1)}, v^{(1)})\|_{\ell^2(\rho)} = \mathcal{O}(\delta^{2p+5}), \quad (2.19)$$

$$\|\Delta \omega^{(1)}\| \asymp \mathcal{O}(\delta^{2p}), \quad (2.20)$$

$$\left| \det\left(\frac{\partial \omega^{(1)}}{\partial a}\right) \right| \asymp \mathcal{O}(\delta^{2p}). \quad (2.21)$$

Moreover  $\omega^{(1)}$  is Diophantine

$$\|n \cdot \omega^{(1)}\|_{\mathbb{T}} \geq \frac{\kappa \delta^{2p}}{|n|^\gamma}, \quad n \in \mathbb{Z}^b \setminus \{0\}, \quad \kappa > 0, \gamma > 2b + 1, \quad (2.22)$$

where  $\|\cdot\|_{\mathbb{T}}$  denotes the distance to integers in  $\mathbb{R}$ ,  $\kappa = \kappa(\epsilon)$  and  $\gamma$  are independent of  $\delta$ .

*Proof.* From the Newton scheme

$$\begin{aligned} \Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} &= [F'(u^{(0)}, v^{(0)})]^{-1} F(u^{(0)}, v^{(0)}) \\ &= [D'^{-1} + F'^{-1}(A - \delta^{2p} \text{diag } A_0) D'^{-1}] F, \end{aligned}$$

where  $\text{diag } A_0$  is the diagonal part of  $A_0$ ,  $D' = D + \delta^{2p} \text{diag } A_0$ ,  $D$  and  $A_0$  as in (1.14, 1.15).

Let  $F_0(u^{(0)}, v^{(0)})$  be as in (1.31),  $(\Delta n, \Delta j) \in \Gamma^{++}$  defined in (1.16) and  $(-e_m, j_m) \in \text{supp } u^{(0)}$ . If  $(\Delta n, \Delta j) = (-e_{k'} + e_m, j_{k'} - j_m)$  for some  $k' = 1, \dots, b$ ,

$$(-e_m, j_m) + (\Delta n, \Delta j) = (-e_{k'}, j_{k'}) \in \mathcal{S}.$$

Otherwise (iv) gives

$$(-e_m, j_m) + (\Delta n, \Delta j) \notin \mathcal{C}.$$

Therefore

$$\text{supp } F_0(u^{(0)}, v^{(0)}) \cap \{\mathcal{C} \setminus \mathcal{S}\} = \emptyset,$$

$$\|D'^{-1}F\|_{\ell^2} = \mathcal{O}(\delta^3)$$

and

$$\|F'^{-1}(A - \delta^{2p} \text{diag } A_0)D'^{-1}F\|_{\ell^2} = \mathcal{O}(\delta^3),$$

where we also used (2.2). So

$$\|\Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}\|_{\ell^2} = \mathcal{O}(\delta^3)$$

and

$$\|F(u^{(1)}, v^{(1)})\|_{\ell^2} \leq \mathcal{O}(\|F''\|) \|\Delta \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}\|_{\ell^2}^2 = \mathcal{O}(\delta^{2p+5}).$$

Using (2.3), the above two estimates hold in weighted space as well and we obtain (2.18, 2.19).

From the  $Q$  equations, the frequency modulation

$$\Delta \omega_j^{(1)} = \frac{1}{a_j} F(u^{(0)}, v^{(0)})(-e_j, j)$$

is a rational function in  $a$  of degree  $p+1$  for each  $j$ . So  $\|\Delta \omega^{(1)}\|^2$  is controlled from below by a polynomial in  $a$  of degree  $2(p+b)$ . Using the same argument as in (2.16), we obtain (2.20).

Similarly  $\det(\frac{\partial \omega^{(1)}}{\partial a})$  is a rational function in  $a$  of degree at most  $(p+1)b$ , and is controlled from below by a polynomial of degree at most  $pb + b + 2b! - 2$ . So (2.16) gives (2.21), which in turn implies the Diophantine property (2.22) for appropriate  $\kappa$  as  $d^b a = \frac{d^b \omega^{(1)}}{|\det(\frac{\partial \omega^{(1)}}{\partial a})|}$  and the excised set in  $\omega^{(1)}$  is of measure  $\mathcal{O}(\delta^{2p})$ .  $\square$

### 3. The second step

Proposition 2.2, in particular (2.22) has transformed the (tangentially) resonant perturbation into non resonant perturbation by amplitude-frequency modulation. However due to the extra factor of  $\delta^{2p}$ , we need to proceed differently at the initial step. Essentially we redo Lemma 2.1 at scale  $N = |\log \delta|^s$  ( $s > 1$ ) with Diophantine  $\omega^{(1)}$  replacing integer  $\omega^{(0)}$ . This is made possible by the fact that the resonance structure remains the same at this scale.

Define the truncated linearized operator  $F'_N(u^{(1)}, v^{(1)})$  evaluated at  $\omega^{(1)}$  as

$$\begin{cases} F'_N(u^{(1)}, v^{(1)})(x, y) &= F'(u^{(1)}, v^{(1)})(x, y), & \|x\|_\infty \leq N, \|y\|_\infty \leq N \\ &= 0, & \text{otherwise.} \end{cases} \quad (3.1)$$

We have the analogue of Lemma 2.1.

**Lemma 3.1.** *Assume Proposition 2.2 holds. There exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exists  $\delta_0 > 0$  and for all  $\delta \in (0, \delta_0)$  there exists a set  $\tilde{\mathcal{B}}_\epsilon \supset \mathcal{B}_\epsilon$  with*

$$\text{meas } (\tilde{\mathcal{B}}_\epsilon \cap \mathcal{B}) < C\epsilon^c,$$

where  $\mathcal{B} = (0, 1]^b$ . If  $a \in \mathcal{B} \setminus \tilde{\mathcal{B}}_\epsilon$ , then

$$\|[F'_N(u^{(1)}, v^{(1)})]^{-1}\| \leq \mathcal{O}(\delta^{-2p-\epsilon}) \quad (3.2)$$

and there exists  $\beta \in (0, 1)$  such that

$$|[F'_N(u^{(1)}, v^{(1)})]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (3.3)$$

for all  $|x - y| > 1/\beta^2$ .

(3.2, 3.3) together with (2.22) put the construction in the standard non resonant form as put forward by Bourgain [B2, 3], cf. also [BW]. (2.21) ensures transversality, in other words, continuing amplitude-frequency modulation. The  $\delta^{2p}$  factor in (2.22) is compensated by  $\delta^{2p+5}$  in the remainder in (2.19).

*Proof of Lemma 3.1.* Using (2.18, 2.19), it suffices to work with  $F'_{0,N}(u^{(0)}, v^{(0)})$  (evaluated at  $\omega^{(1)}$ ). The geometric construction in the proof of Lemma 2.1 remains valid:

$$PF'_{0,N}P = \oplus_k \Gamma_k(a), \quad k \leq K_1(N),$$

where  $\Gamma_k$  are matrices of at most sizes  $(2b + d) \times (2b + d)$ ,

$$\Gamma_k = \begin{pmatrix} \text{diag } (n \cdot \Delta\omega^{(1)}) & 0 \\ 0 & \text{diag } (-n \cdot \Delta\omega^{(1)}) \end{pmatrix} + \delta^{2p} \mathcal{A}_k,$$

and  $\mathcal{A}_k$  are convolution matrices.

With the addition of the diagonal term,  $\Gamma_k$  is no longer a convolution matrix. Moreover  $|n| \leq N = |\log \delta|^s$  ( $s > 1$ ) depends on  $\delta$ . So we need to proceed differently because of uniformity considerations in estimates of type (2.16).

Fix

$$N_0 = 100(2b + d). \quad (3.4)$$

For a given  $\Gamma_k$ , define the support of  $\Gamma_k$  to be

$$\mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d} \supset \text{supp } \Gamma_k = \{(x, y) | \Gamma_k(x, y) \neq 0\}.$$

For matrices  $\Gamma_k$ , such that

$$\text{supp } \Gamma_k \cap \{[-N_0, N_0]^{b+d} \times [-N_0, N_0]^{b+d}\} \neq \emptyset, \quad (3.5)$$

we proceed as in the proof of Lemma 2.1. There are at most  $K_0$  (independent of  $\delta$ ) of these matrices. Let  $P_k = P_k(a) = \det \Gamma_k(a)$ , we have that there exist  $C, c > 0$ , such that for all  $0 < \epsilon < 1$ ,

$$\text{meas } \{a \in \mathcal{B} | |P_k| < \delta^{2p+\epsilon}, \text{ all } k \leq K_0\} \leq C\delta^{c\epsilon}.$$

So  $\|\Gamma_k^{-1}\| \leq \mathcal{O}(\delta^{-2p-\epsilon})$  for all  $k \leq K_0$ . It is important to note that  $C, c$  are *independent* of  $\delta$ , since  $P_k/\delta^{2p} = P'_k + \mathcal{O}(\delta^2)$  and  $P'_k$  is *independent* of  $\delta$ .

For matrices  $\Gamma_k$  with  $k > K_0$ ,

$$\text{supp } \Gamma_k \cap \{[-N_0, N_0]^{b+d} \times [-N_0, N_0]^{b+d}\} = \emptyset$$

by definition. We use perturbation theory. For any  $\Gamma_k$ , fix  $N'$ , with  $|N'| > N_0$ , such that for all  $(n, j) \in \text{supp } \Gamma_k$ , we can write  $(n, j) = (N', 0) + (n', j)$  with  $|n'| \leq (2b + d)$ .

Let  $\lambda$  be an eigenvalue of  $\Gamma_k$  with normalized eigenfunction  $\phi$ . Then

$$\begin{aligned} \lambda &= (\phi, \lambda\phi) = (\phi, [\text{diag } (n \cdot \Delta\omega^{(1)}) + \delta^{2p}\mathcal{A}_k]\phi) \\ &= \sum_{(n,j)} n \cdot \Delta\omega^{(1)} |\phi_{n,j}|^2 + \delta^{2p}(\phi, \mathcal{A}_k\phi) \\ &= N' \cdot \Delta\omega^{(1)} + \sum_{(n',j)} n' \cdot \Delta\omega^{(1)} |\phi_{n,j}|^2 + \delta^{2p}(\phi, \mathcal{A}_k\phi) \end{aligned}$$

First order eigenvalue variation gives

$$N' \cdot \frac{\partial \lambda}{\partial \omega^{(1)}} = N'^2 + \mathcal{O}\left(\frac{N'^2}{100}\right).$$

So

$$\frac{N' \cdot \frac{\partial \lambda}{\partial \omega^{(1)}}}{|N'|} \geq \frac{99}{100} |N'| > 1.$$

Using  $|\det(\frac{\partial \omega^{(1)}}{\partial a})| \asymp \delta^{2p}$  from (2.14) and taking  $\lambda = 0$ , this gives  $\|\Gamma_k^{-1}\| \leq \mathcal{O}(\delta^{-2p-\epsilon})$  for all  $K_0 < k \leq K_1(N)$  away from a set of  $a$  of measure less than  $\delta^{\epsilon/2}$  as  $K_1(N) \leq \mathcal{O}(|\log \delta|^{2s})$ ,  $s > 1$ .

The estimates on  $\Gamma_k$  above and the Schur reduction (2.5) give (3.2). The pointwise estimates follow as in the proof of (2.3), since the geometry of the resonant structure remains the same. Here we also used (2.18).  $\square$

We summarize the findings in Proposition 2.2 and Lemma 3.1 in the following amplitude-frequency modulation proposition. For simplicity, we use  $u$  to denote both the function and its Fourier series as it should be clear from the context.

**Proposition 3.2.** *Assume*

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{ij_k \cdot x} e^{-ij_k^2 t},$$

*a solution to the linear Schrödinger equation (1.2) is generic,  $a = \{a_k\}_{k=1}^b \in (0, 1]^b = \mathcal{B}$ ,  $j_k \neq 0$ ,  $k = 1, \dots, b$  and  $N = |\log \delta|^s$  ( $s > 1$ ). Then there exist  $C, c > 0$ , such that for all  $\epsilon \in (0, 1)$ , there exist  $\delta_0 > 0$  and for all  $\delta \in (0, \delta_0)$  a set  $\tilde{\mathcal{B}}_\epsilon \supset \mathcal{B}_\epsilon$  with*

$$\text{meas } (\tilde{\mathcal{B}}_\epsilon \cap \mathcal{B}) < C\epsilon^c.$$



If  $a \in \mathcal{B} \setminus \tilde{\mathcal{B}}_\epsilon$ , then

$$\|\Delta u^{(1)}\|_{\ell^2(\rho)} = \|\Delta v^{(1)}\|_{\ell^2(\rho)} = \mathcal{O}(\delta^3),$$

where  $\rho$  is a weight on  $\mathbb{Z}^{b+d}$  satisfying

$$\begin{aligned} \rho(x) &= e^{\beta|x|}, \quad 0 < \beta < 1 \text{ for } |x| > x_0 \\ &= 1, \quad \text{for } |x| \leq x_0. \end{aligned}$$

$$\|F(u^{(1)}, v^{(1)})\|_{\ell^2(\rho)} = \mathcal{O}(\delta^{2p+5}),$$

$$\|\Delta \omega^{(1)}\| \asymp \mathcal{O}(\delta^{2p}),$$

$$\left| \det\left(\frac{\partial \omega^{(1)}}{\partial a}\right) \right| \asymp \mathcal{O}(\delta^{2p}),$$

and  $\omega^{(1)}$  is Diophantine

$$\|n \cdot \omega^{(1)}\|_{\mathbb{T}} \geq \frac{\kappa \delta^{2p}}{|n|^\gamma}, \quad n \in \mathbb{Z}^b \setminus \{0\}, \quad \kappa > 0, \gamma > 2b + 1,$$

where  $\|\cdot\|_{\mathbb{T}}$  denotes the distance to integers in  $\mathbb{R}$ ,  $\kappa = \kappa(\epsilon)$  and  $\gamma$  are independent of  $\delta$ .

Moreover

$$\|[F'_N(u^{(1)}, v^{(1)})]^{-1}\| \leq \mathcal{O}(\delta^{-2p-\epsilon}) \quad (3.6)$$

and

$$|[F'_N(u^{(1)}, v^{(1)})]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (3.7)$$

for all  $|x - y| > 1/\beta^2$ , where  $F'_N$  as in (3.1) with  $N = |\log \delta|^s$  ( $s > 1$ ).

#### 4. Proof of the Theorem

Proposition 3.2 provides the input for the initial scale in the Newton scheme in [B3]. To continue the iteration, we need the analogues of (3.6, 3.7) at larger scales. This is attained as follows.

Let  $T = F'$  be the linearized operator defined as in (1.13-1.15) and the restricted operator  $T_N = F'_N$  as in (3.1). To increase the scale from  $N$  to a larger scale  $N_1$ , we pave the  $N_1$  cubes with  $N$  cubes. In the  $j$  direction, this is taken care of by perturbation; while in the  $n$  direction by adding an additional parameter  $\theta \in \mathbb{R}$  and consider  $T(\theta)$ :

$$T(\theta) = \begin{pmatrix} \text{diag}(n \cdot \omega + j^2 + \theta) & 0 \\ 0 & \text{diag}(-n \cdot \omega + j^2 - \theta) \end{pmatrix} + \delta^{2p} A,$$

where  $A$  as defined in (1.15).

Let  $N = |\log \delta|^s$  ( $s > 1$ ) as in Lemma 3.1 and  $T_N(\theta) = T_N(\theta; u^{(1)}, v^{(1)})$  evaluated at  $\omega^{(1)}$ . We have the following estimates.

**Lemma 4.1.** Assume  $u^{(0)} = \sum_{k=1}^b a_k e^{ij_k \cdot x} e^{-ij_k^2 t}$  a solution to the linear Schrödinger equation (1.2) is generic and  $a \in \mathcal{B} \setminus \tilde{\mathcal{B}}_\epsilon$ , the set defined in Proposition 3.2. Then

$$\|[T_N(\theta)]^{-1}\| \leq \mathcal{O}(\delta^{-2p-\epsilon}) \quad (4.1)$$

and there exists  $\beta \in (0, 1)$  such that

$$|[T_N(\theta)]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (4.2)$$

for all  $|x - y| > 1/\beta^2$ , away from a set  $B_N(\theta) \subset \mathbb{R}$  with

$$\text{meas } B_N(\theta) < \delta^{2p+c\epsilon}.$$

*Proof.* For any  $\theta \in \mathbb{R}$ , we can always write  $\theta = \Theta + \delta^{2p}\theta'$ , where  $\Theta \in \mathbb{Z}$ ,  $-\frac{1}{2} \leq \delta^{2p}\theta' < \frac{1}{2}$ . Since  $\|T_N\| \leq 2|\log \delta|^{2s}$  ( $s > 1$ ), we may restrict  $\Theta$  to  $|\Theta| \leq 2|\log \delta|^{2s} + 1$ .

Then

$$\begin{aligned} T_N(\theta) = & \begin{pmatrix} \text{diag } (n \cdot \omega^{(0)} + j^2 + \Theta) & 0 \\ 0 & \text{diag } (-n \cdot \omega^{(0)} + j^2 - \Theta) \end{pmatrix} \\ & + \delta^{2p} \begin{pmatrix} \text{diag } (n \cdot \tilde{\omega} + \theta') & 0 \\ 0 & \text{diag } (-n \cdot \tilde{\omega} - \theta') \end{pmatrix} + \delta^{2p} A_N, \end{aligned}$$

where  $\omega^{(0)} \in \mathbb{Z}^b$ ,  $\Theta \in \mathbb{Z}$ ,  $\tilde{\omega} = \Delta\omega^{(1)}/\delta^{2p}$  is Diophantine and  $A_N$  is the restricted  $A$  as defined in (1.15, 3.1).

Let

$$\mathcal{H} = \delta^{2p} \left[ \begin{pmatrix} \text{diag } (n \cdot \tilde{\omega} + \theta') & 0 \\ 0 & \text{diag } (-n \cdot \tilde{\omega} - \theta') \end{pmatrix} + A_N \right].$$

Let  $P_+$  be the projection onto the set  $\{(n, j) | n \cdot \omega^{(0)} + j^2 + \Theta = 0\}$  and  $P_-$  the projection onto the set  $\{(n, j) | -n \cdot \omega^{(0)} + j^2 - \Theta = 0\}$  when  $\Theta \neq 0$ ; when  $\Theta = 0$ , use the definition in the proof of Lemma 2.1. Define

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$$

and  $P^c$  the projection onto the complement as before.

We proceed using the Schur reduction as in the proof of Lemme 2.1 and 3.1. It suffices to estimate  $P\mathcal{H}P = \oplus_k \Gamma_k(\theta)$  as  $P^c T_N P^c$  is invertible,  $\|(P^c T_N P^c - \lambda)^{-1}\| \leq 4$  uniformly in  $\theta$  and  $\lambda \in [-1/4, 1/4]$ .

For  $\Gamma_k$  such that (3.5) is satisfied, we obtain

$$\text{meas } \{\theta | \|\Gamma_k^{-1}\| > \delta^{-2p-\epsilon}\} \leq C\delta^{c\epsilon} \quad (\epsilon > 0),$$

via the estimates on  $\det \Gamma_k$  as in the proof of Lemma 3.1. For other  $\Gamma_k$ , we make a shift  $n \cdot \tilde{\omega} = N' \cdot \tilde{\omega} + n' \cdot \tilde{\omega}$ , where  $|N'| < |\log \delta|^s$ ,  $|n'| < N_0$ ,  $N_0$  as in (3.4). Let  $\theta'' = \theta' + N' \cdot \tilde{\omega}$ . Then clearly

$$\text{meas } \{\theta'' | \text{dist } (\theta'', \sigma(\Gamma^k)) \leq \delta^{2p+\epsilon}\} \leq \mathcal{O}(\delta^{2p+\epsilon}).$$

Combining the above estimates on  $\Gamma_k$ , we obtain (4.1). Using the geometric structure afforded by the resonance structure, we obtain (4.2) as before.  $\square$

*Proof of the Theorem.* This follows from Lemma 4.1, Proposition 3.2, lemme 19.13, 19.38 and 19.65 in Chapter 19 of [B3].  $\square$

*Proof of the Corollary.* The generic conditions (i-iv) lead to homogeneous polynomials. So if  $\{j_k\}_{k=1}^b$  is generic, then so is  $\{Kj_k\}_{k=1}^b$  for all  $K \in \mathbb{N} \setminus \{0\}$ . So we may assume  $j_k = \mathcal{O}(1)$  ( $k = 1, \dots, b$ ). Generalizing the definition of bi-characteristics to

$$\mathcal{C} = \{(n, j) \in \mathbb{Z}^{b+d} \mid |\pm n \cdot \omega^{(0)} + j^2| \leq \mu\},$$

for some  $\mu > \|A\|$  independent of  $K$ , where  $A$  is the convolution operator generated by the nonlinear term as in (1.13), and modifying the sets  $\{\Sigma_{\pm}\}$ ,  $\{J\}$  and  $\{f\}$  to  $\{\cdot\} + \{0, \pm 1, \dots, \pm \mu\}$  respectively, (i-iv) remain valid for sufficiently large  $K$ .

Now if  $(n, j) \in \mathcal{C}$ , then by definition

$$[\pm n \cdot \omega^{(0)} + j^2] \in \{0, \pm 1, \dots, \pm \mu\} := M,$$

where  $\omega^{(0)} = \{K^2 j_k^2\}_{k=1}^b$ . So  $n \cdot \omega^{(0)} \in K^2 \mathbb{Z}$ , and  $j^2 \in K^2 \mathbb{Z} + M$ . Since  $\Delta n \in \mathbb{Z}^b$  and  $\Delta j \in K \mathbb{Z}^d$  from (1.16, 1.17), this gives

$$\pm(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2 \in K \mathbb{Z} + M.$$

So if

$$[\pm(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2] \notin M,$$

then

$$|\pm(n + \Delta n) \cdot \omega^{(0)} + (j + \Delta j)^2| \geq K - \mu > K/2,$$

for large  $K$ .

Using this to estimate the second term in (2.5) gives that the linearized operator  $\|F'^{-1}\| \leq \mathcal{O}(1)/\epsilon$  for the initial step. Now  $|f| \geq K^2 - \mu$  from (iv), which gives  $\|\Delta u\| \leq \mathcal{O}(1)/K^2$  for the first iteration. Taking  $\mu = |\log \epsilon|^s$  ( $s > 1$ ) and the initial scale  $N = |\log \epsilon|^{s'}$  in the  $n$  direction and  $N = K|\log \epsilon|^{s'}$  in the  $j$  direction with  $1 < s' < s$ , we obtain the analog of Proposition 3.2. The proof of the Corollary then proceeds as that of the Theorem.  $\square$

## 5. Appendix: the cubic nonlinearity

For simplicity we write  $u$  for  $u^{(0)}$  and  $\omega$  for  $\omega^{(0)}$ , the solutions and frequencies of the linear equation. The symbols of convolution for the cubic nonlinearity are  $|u|^2$ ,  $u^2$  and  $\bar{u}^2$ . Assume  $(n, j) \in \mathcal{C}^+$  ( $(n, j) \in \mathcal{C}^-$  works similarly). In order that  $(n, j)$  is connected to  $(n', j') \in \mathcal{C}$ , it is necessary that either

- (a)  $[u * v](n, j; n' j') \neq 0$  or
- (b)  $[u * u](n, j; n' j') \neq 0$ .

Case (a): Since

$$\begin{aligned} n \cdot \omega + j^2 &= 0, \\ n' \cdot \omega + j'^2 &= 0, \end{aligned}$$

subtracting the two equations gives immediately

$$(j_k - j'_k) \cdot (j + j_k) = 0, \quad (5.1)$$

where  $j_k, j_{k'} \in \mathbb{Z}^d$  ( $k, k' = 1, \dots, b$ ) and  $j_k \neq j_{k'}$  if  $k \neq k'$ , are the  $b$  Fourier components of  $u$ .

Case (b): Since

$$\begin{aligned} n \cdot \omega + j^2 &= 0, \\ -n' \cdot \omega + j'^2 &= 0, \end{aligned}$$

adding the two equations gives immediately

$$(j + j_k) \cdot (j + j_{k'}) = 0, \quad (5.2)$$

where  $j_k, j_{k'} \in \mathbb{Z}^d$  ( $k, k' = 1, \dots, b$ ) and  $j_k \neq j_{k'}$  if  $k \neq k'$ , are the  $b$  Fourier components of  $u$ .

(5.1, 5.2) are precisely the well known resonant set for the partial Birkhoff normal form transform in [KP, B2, GXY]. (5.1, 5.2) describe rectangular type of geometry.

$$\text{supp } F_0(u, v) \cap \{\mathcal{C} \setminus \mathcal{S}\} = \emptyset$$

for the cubic nonlinearity in any  $d$ . When  $d = 1$ , (5.1, 5.2) reduce to a finite set of  $2b$  lattice points in  $\mathbb{Z}$ :  $\{j = \pm j_k, k = 1, \dots, b\}$  and  $\Omega = \emptyset$  in the Theorem.

## REFERENCES

- [B1] J. Bourgain, *Fourier transformation restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Schrödinger equations*, Geom. and Func. Anal. **3** (1993), 107-156.
- [B2] J. Bourgain, *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math. **148** (1998), 363-439.
- [B3] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*, Ann. of Math. Studies **158** (2005), Princeton University Press.
- [BW] J. Bourgain, W.-M. Wang, *Quasi-periodic solutions of nonlinear random Schrödinger equations*, J. Eur. Math. Soc. **10** (2008), 1-45.
- [CW] W. Craig, C. E. Wayne, *Newton's method and periodic solutions of nonlinear equations*, Commun. Pure Appl. Math. **46** (1993), 1409-1498.
- [EK] L. H. Eliasson, S. E. Kuksin, *KAM for the nonlinear Schrödinger equation*, to appear Ann. of Math.
- [FS] J. Fröhlich, T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), 151-184.

- [GXY] J. Geng, X. Xu, J. You, *Quasi-periodic solutions for two dimensional cubic Schrödinger equation without the outer parameter*, Maiori workshop (2009).
- [GS] M. Goldstein, W. Schlag, *Hölder continuity of the integrated density of states for quasi-periodic Schrödinger operators and averages of shifts of subharmonic functions*, Ann. of Math. **154** (2001), 155-203.
- [G] M. Grassin, *Global smooth solutions to Euler equations for a perfect gas*, Indiana Univ. Math. J. **47** (1998), 1397-1432.
- [KP] S. Kuksin, J. Pöschel, *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math. **143** (1996), 149-179.
- [S1] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, I*, J. Reine Angew. Math. **147** (1917), 205-232.
- [S2] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, II*, J. Reine Angew. Math. **148** (1918), 122-145.
- [Se] D. Serre, *Solution classique globales des équations d'Euler pour un fluide parfait compressible*, Ann. Inst. Fourier **47** (1997), 139-153.
- [W1] W.-M. Wang, *Bounded Sobolev norms for linear Schrödinger equations under resonant perturbations*, J. Func. Anal. **254** (2008), 2926-2946.
- [W2] W.-M. Wang, *Eigenfunction localization for the 2D periodic Schrödinger operator*, Int. Math. Res. Notes (to appear) (2010).

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ PARIS SUD, 91405 ORSAY CEDEX, FRANCE

*E-mail address:* wei-min.wang@math.u-psud.fr